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CONFIDENCE INTERVALS FOR THE COMMON VARIANCE OF EQUICORRELATED --ETC(U)
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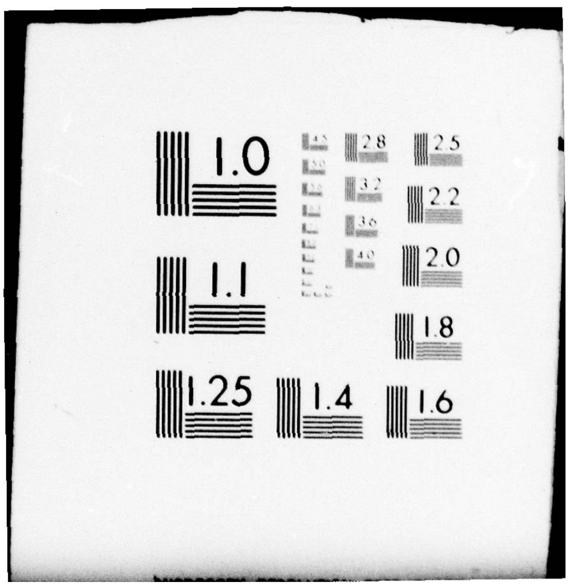
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CONFIDENCE INTERVALS FOR THE COMMON VARIANCE OF EQUICORRELATED NORMAL RANDOM VARIABLES.

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by

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1. INTRODUCTION

The pooling of variance estimates, obtained under different experimental conditions, is a classical problem in statistical inference. The commonly used formulas for pooling in the normal case are based on the assumption that the samples are independent and the population variances are equal or their ratios are known. Previous studies were mainly oriented to the pooling problem when the assumption about the equality of variances is questionable. However, little is known concerning the situation when the samples may not be independent. In the present study we address the problem of estimating the common variance of several normal populations which may be correlated. More specifically we consider a model of several independent replicas from a multivariate normal distribution having a covariance matrix of the form $\sigma^2 \underset{\sim}{R}$ where $\underset{\sim}{R}$ is a correlation matrix for equicorrelated random variables, with unknown variance, σ^2, and correlation coefficient, ρ. To simplify we present the development under the assumption that the means are known. Straightforward generalization to the case of unknown means can be given for practical applications by substituting their maximum likelihood estimates and modifying the number of degrees of freedom. However, the theoretical problem of admissibility of the maximum likelihood procedures, as indicated by Stein [7], arises and Stein type estimators can be attempted.

We investigate the problem of determining efficient confidence intervals for ρ and σ^2. Confidence intervals for ρ were previously determined by Olkin and Pratt [6]. Nontrivial confidence intervals for σ^2, when ρ is unknown, are more difficult to obtain. There exists no uniformly most

accurate invariant (UMAI) confidence interval for σ^2 since ρ is a nuisance paramter. In the present study we develop interval estimators for σ^2, which are asymptotically most efficient, by employing the best asymptotic normality, (BAN), of the maximum likelihood estimators (see Zacks [11] p.244). The problem is to evaluate the coverage probabilities of these MLE based interval estimators in small or medium size samples. Exact coverage probabilities were determined by employing distributional properties of the statistics involved and some numerical integrations. The results indicate that the actual coverage probabilities of the intervals proposed are close to the nominal ones, when the number of replicas is at least 20. For smaller samples the coverage probabilities may show greater deviation from the nominal ones, especially when $|\rho|$ is close to one. We have therefore investigated the properties of alternative estimators in small sample situations.

When ρ is known one can construct UMAI confidence intervals for σ^2. The statistic used in this case suggests a system of confidence intervals in which proper estimates are substituted for the unknown ρ. This type of intervals will be called "estimated-ρ intervals". In order to secure the required coverage probability one may attempt to apply the Bonferroni inequality and use confidence limits for ρ, rather than point estimates. We show that such an approach results in very inefficient confidence intervals. By trial and error the development of more efficient intervals of this type might be possible. More research should be performed in order to derive universally good estimated-ρ type confidence intervals. In addition, we

compare the MLE based confidence intervals with "naive" type of intervals, which employs in a non-optimal fashion the information on σ^2 obtained from the individual samples. We show in the present paper that despite the slight small sample deficiencies of the MLE based confidence intervals, they are generally more efficient than the other types of intervals considered here. For this purpose we introduce a measure of relative efficiency which is based on the actual coverage probability and the expected length of the intervals.

The determination of the actual coverage probabilities and the expected length is generally a difficult matter. We provide a method for exact numerical determination in the case of the MLE based intervals. For the "naive" type intervals an explicit formula for the expected length has been derived. For evaluating the efficiency of the "estimated-ρ type intervals" we have employed the Monte Carlo method. Finally, we have investigated the loss in actual coverage incurred by ignoring the possibility of correlation and employing the usual confidence intervals for the case of $\rho = 0$.

There are many practical problems for which the above model of equicorrelated homoscedastic observations applies. For example, the compressive strength of concrete cubes from the same batch are generally correlated. Since a batch is homogeneously mixed, all the cubes from the same batch will have strength values with equal variances and the correlation between any pair of cubes will be the same. Cubes prepared from different batches will generally be independent and will have the same variance if the manufacturing conditions are similar.

A frequent application of the equicorrelated, equal variance model is outlined by Winer [9] for the one-way repeated measures design.

2. THE MODEL, THE LIKELIHOOD FUNCTION AND THE FISHER INFORMATION

Let $\underset{\sim}{X}$ be an mx1 random vector having an equicorrelated multinormal distribution with zero mean vector and covariance matrix

$$\Sigma = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{J}], \tag{2.1}$$

$0 < \sigma^2 < \infty$, $-(m-1)^{-1} < \rho < 1$; where $m \geq 2$, and $\underset{\sim}{J} = \underset{\sim}{1}\underset{\sim}{1}'$ where $\underset{\sim}{1}' = (1,1,\ldots,1)$. Consider the Helmert transformation $\underset{\sim}{Y} = \underset{\sim}{H}\underset{\sim}{X}$ where $\underset{\sim}{H}$ is an orthogonal matrix with first row vector equal to $m^{-\frac{1}{2}}\underset{\sim}{1}'$ (see Kendall and Buckland [3] p.126). The components of $\underset{\sim}{Y}$ are independent normal random variables, with zero means and variances

$$\operatorname{Var}(Y_j) = \begin{cases} \sigma^2(1 + (m-1)\rho) & j = 1 \\ \sigma^2(1-\rho) & j = 2,\ldots,m. \end{cases} \tag{2.2}$$

Given n independent identically distributed observations on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$, the likelihood function of (σ^2,ρ) can be expressed in terms of the $\underset{\sim}{Y}$ vectors as

$$L(\sigma^2,\rho;\ \underset{\sim}{Y}_1,\underset{\sim}{Y}_2,\ldots,\underset{\sim}{Y}_n) = c(\sigma^2)^{-mn/2}(1 + (m-1)\rho)^{-n/2}\,.$$
$$(1-\rho)^{-(m-1)n/2}\exp\{-(2\sigma^2)^{-1}[(1 + (m-1)\rho)^{-1}\sum_{i=1}^{n} Y_{i1}^2 \tag{2.3}$$
$$+ (1-\rho)^{-1}\sum_{i=1}^{n}\sum_{j=2}^{m} Y_{ij}^2]\}.$$

Define $Q_1 = \sum_{i=1}^{n} Y_{i1}^2$ and $Q_2 = \sum_{i=1}^{n}\sum_{j=2}^{m} Y_{ij}^2$.

It follows that Q_1 and Q_2 are independent and constitute complete sufficient atatistics for the present model (see Lehmann [4] p.132).

Furthermore, let $\chi^2[\nu]$ designate a chi-squared r.v. with ν degrees of freedom, then

$$Q_1 \sim \sigma^2(1 + (m-1)\rho)\ \chi^2[n]$$

and (2.4)

$$Q_2 \sim \sigma^2(1-\rho)\ \chi^2\ [n(m-1)].$$

The log-likelihood (sample information) function can then be expressed as

$$\ell(\sigma^2,\rho;\ \underset{\sim}{Y}_1,\underset{\sim}{Y}_2,\ldots,\ \underset{\sim}{Y}_n) = \log c - \frac{mn}{2}\log\sigma^2 - \frac{n}{2}\log(1 + (m-1)\rho)$$

$$\frac{-(m-1)n}{2}\ \log(1-\rho) - (2\sigma^2)^{-1}[(1 + (m-1)\rho)^{-1}Q_1 + (1-\rho)^{-1}Q_2]. \tag{2.5}$$

The score functions are given by

$$S_1(\sigma^2,\rho) = \frac{\partial\ell}{\partial\sigma^2} = -mn/(2\sigma^2)+Q_1/[2\sigma^4(1+(m-1)\rho)]+Q_2/[2\sigma^4(1-\rho)]$$

$$S_2(\sigma^2,\rho) = \frac{\partial\ell}{\partial\rho} = -n(m-1)/[2(1+(m-1)\rho)] + n(m-1)/[2(1-\rho)] \tag{2.6}$$

$$+(m-1)Q_1/[2\sigma^2(1+(m-1)\rho)^2]-Q_2/[2\sigma^2(1-\rho)^2]\ .$$

The maximum likelihood estimator of $\underset{\sim}{\theta} = (\sigma^2,\rho)$ is obtained from (2.6) as the roots of the system of equations $S_1(\sigma^2,\rho)\equiv 0$; $S_2(\sigma^2,\rho)\equiv 0$. These MLE's are

$$\hat{\sigma}^2 = (Q_1 + Q_2)/(mn)$$

and (2.7)

$$\hat{\rho} = (Q_1 - (m-1)^{-1}Q_2)/(Q_1 + Q_2).$$

Since the present model satisfies the Cramér-Rao regularity conditions (Wijsman [8]), $E_{\underset{\sim}{\theta}}\ \{S_1(\sigma^2,\rho)\} = 0$ and $E_{\underset{\sim}{\theta}}\ \{S_2(\sigma^2,\rho)\} = 0$ and the variances and covariances of the score functions are given by

$$\begin{aligned}
\mathrm{Var}(S_1) &= nm/(2\sigma^4), \\
\mathrm{Var}(S_2) &= n(m-1)/2\{((m-1)(1-\rho)^2 + (1 + (m-1)\rho)^2) \cdot \\
&\qquad ((1-\rho)^2(1 +(m-1)\rho)^2)^{-1}\} \\
\mathrm{Cov}(S_1,S_2) &= -n(m-1)m\rho/[2\sigma^2(1-\rho)(1 +(m-1)\rho)].
\end{aligned} \tag{2.8}$$

Accordingly, the Fisher information matrix is

$$\underset{\sim}{I}(\sigma^2,\rho) = \begin{bmatrix} \mathrm{Var}(S_1) & \mathrm{Cov}(S_1,S_2) \\ \cdot & \mathrm{Var}(S_2) \end{bmatrix} \tag{2.9}$$

and its inverse is

$$\underset{\sim}{I}^{-1}(\sigma^2,\rho) = \begin{bmatrix} 2\sigma^4(1 +(m-1)\rho^2)/(nm) & 2\sigma^2\rho(1-\rho)(1+(m-1)\rho)/(nm) \\ \cdot & 2(1-\rho)^2(1+(m-1)\rho)^2/(nm(m-1)) \end{bmatrix} \tag{2.10}$$

Hence the asymptotic variance of any asymptotically normal estimator of σ^2 is at least $2\sigma^4(1+(m-1)\rho^2)/(nm)$.

3. POINT AND INVERVAL ESTIMATORS BASED ON THE MLE

The MLE of σ^2 given in (2.7) is uniformly minimum variance unbiased (UMVU) having a variance

$$V(\hat{\sigma}^2) = 2\sigma^4(1+(m-1)\rho^2)/(mn). \tag{3.1}$$

This variance coincides with the asymptotic variance of the MLE, which is presented in $\underset{\sim}{I}^{-1}(\sigma^2,\rho)$. Furthermore, $\hat{\sigma}^2$ is best asymptotically normal estimator (BAN).

Since Q_1 and Q_2 have gamma distributions with different scale parameters, their sum does not have a gamma distribution. The exact distribution of $\hat{\sigma}^2$ can be expressed as a mixture of gamma distributions (see Neuts and Zacks [5]). This distribution is, however, too complicated to provide a simple system of confidence intervals when ρ is unknown. We therefore construct interval estimators, which are based on the asymptotic normality of the MLE's with the aim of attaining a $(1-\alpha)$ coverage probability. These interval estimators are based on the fact that when n is large then

$$P\{\left|\frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2 \delta(\hat{\rho})}\right| \le z_\gamma\} \cong 1 - \alpha \tag{3.2}$$

where $\gamma = 1 - \alpha/2$, z_γ is the γ-fractile of the standard normal distribution and

$$\delta(\hat{\rho}) = [2(1 + (m-1)\hat{\rho}^2)/mn]^{\frac{1}{2}}. \tag{3.3}$$

From (3.2) one obtains the system of interval estimators of the form $(\hat{\sigma}^2/b_\gamma(\hat{\rho}), \hat{\sigma}^2/a_\gamma(\hat{\rho}))$, where $a_\gamma(\hat{\rho}) = \max(0, 1 - z_\gamma\,\delta(\hat{\rho}))$ and $b_\gamma(\hat{\rho}) = 1 + z_\gamma\,\delta(\hat{\rho})$. Notice that in small samples $z_\gamma\,\delta(\hat{\rho})$ could exceed one with positive probability. This is not the problem in large samples since, as follows from (3.3), $\delta(\hat{\rho}) \le (2/n)^{\frac{1}{2}}$. Thus, in small samples, the above system may have a very large or infinite expected length. In order to overcome this difficulty, we propose a modified system of MLE based interval estimators, namely $(\hat{\sigma}^2 A_\gamma(\hat{\rho}), \hat{\sigma}^2 B_\gamma(\hat{\rho}))$, where $A_\gamma(\hat{\rho}) = \max(0, 1 - t_\gamma[\nu_{\hat{\rho}}]\ \delta(\hat{\rho}))$ and $B_\gamma(\hat{\rho}) = 1 + t_\gamma[\nu_{\hat{\rho}}]\ \delta(\hat{\rho})$. $t_\gamma[\nu_{\hat{\rho}}]$ is the γ-fractile of the t-distribution with $\nu_{\hat{\rho}}$ degrees of freedom where $\nu_{\hat{\rho}}$ is taken as the integer part of $n(1 + (m-1)(1-\hat{\rho}^2))$. We have replaced z_γ by $t_\gamma[\nu_{\hat{\rho}}]$ in order to compensate for interchanging the roles of $\hat{\sigma}^2$ and σ^2 in (3.2). Thus, the

MLE based confidence intervals are of the form

$$(\hat{\sigma}^2/A_\gamma^*(\hat{\rho}),\ \hat{\sigma}^2/B_\gamma^*(\hat{\rho})) \tag{3.4}$$

where

$$A_\gamma^*(\hat{\rho}) = \begin{cases} b_\gamma(\hat{\rho}) \ , & \text{MLE intervals} \\ 1/A_\gamma(\hat{\rho}) \ , & \text{Modified-MLE intervals} \end{cases} \tag{3.5}$$

$$B_\gamma^*(\hat{\rho}) = \begin{cases} a_\gamma(\hat{\rho}) \ , & \text{MLE intervals} \\ 1/B_\gamma(\hat{\rho}) \ , & \text{Modified-MLE intervals.} \end{cases} \tag{3.6}$$

We conclude the present section with a comment on confidence intervals for ρ. From (2.4) it is immediately implied that

$$\frac{Q_1}{Q_2} \sim \frac{1+(m-1)\rho}{1-\rho} \cdot \frac{n}{n(m-1)} F[n, n(m-1)] \tag{3.7}$$

where $F[\nu_1, \nu_2]$ designates an F-statistic with ν_1 and ν_2 degrees of freedom. UMAI confidence interval for ρ can be directly obtained from (3.7) (see Lehmann [4]).

4. DETERMINATION OF THE EXACT COVERAGE PROBABILITY AND EXPECTED LENGTH OF THE MLE BASED INTERVALS

From (2.4) and the independence of Q_1 and Q_2 one deduces that

$$\frac{Q_1/(1+(m-1)\rho)}{Q_1/(1+(m-1)\rho) + Q_2/(1-\rho)} \sim \beta(n/2, n(m-1)/2) \tag{4.1}$$

where $\beta(p,q)$ designates a random variable having a beta distribution with

paramters p, q. For the purpose of clearly distinguishing between ρ and $\hat{\rho}$ in what follows, we replace $\hat{\rho}$ by R. According to (2.7)

$$Q_1/(Q_1 + Q_2) = (1 + (m-1) R)/m \tag{4.2}$$

and it follows that the left hand side of (4.1) is equivalent to

$$\varphi(R; \rho) = \frac{(1-\rho)(1 + (m-1)R)}{(1-\rho)(1 + (m-1)R) + (1 + (m-1)\rho)\ (1-R)} \ . \tag{4.3}$$

Accordingly, the probability distribution of R, for each ρ, can be determined from (4.1) and (4.3). In addition, it is easy to verify that $P(R < -(m-1)^{-1}) = 0$.

Consider the random variable

$$Q(\rho) = (1 + (m-1)\rho)^{-1} Q_1 + (1-\rho)^{-1} Q_2 .$$

The distribution of $Q(\rho)$ is like that of $\sigma^2 \chi^2[mn]$.

Algebraic manipulations lead to the expression

$$\begin{aligned} Q(\rho) &= \frac{Q_1 + Q_2}{mn} \cdot \left\{ \frac{n[(1-\rho)(1 + (m-1)R) + (m-1)(1-R)(1 + (m-1)\rho)]}{(1-\rho)\ (1 + (m-1)\rho)} \right\} \\ &= \hat{\sigma}^2\ \Psi(R;\ \rho). \end{aligned} \tag{4.4}$$

That is, $Q(\rho)$ has been factored into the product of the MLE $\hat{\sigma}^2$ and the function $\Psi(R;\ \rho)$. (Notice that for $R \geq -(m-1)^{-1}$, $\Psi(R;\ \rho) \geq 0$.)

We show now that R is independent of $Q(\rho)$. Fix ρ and consider the subfamily of distributions of (Q_1, Q_2), $\mathcal{F}_\rho$, depending on the scale parameter σ^2. The random variable $Q(\rho)$ is a complete sufficient statistic for $\mathcal{F}_\rho$. By Basu's theorem [1] $Q(\rho)$ and R are independent since R is ancillary (invariant with respect to the group of scale transformation).

Hence, $Q(\rho)$ is independent of $\Psi(R;\ \rho)$. It follows that the conditional distribution of $\hat{\sigma}^2\ \Psi(R;\ \rho)$, given R, is like that of the marginal distribution of $Q(\rho)$, namely that of $\sigma^2\chi^2[nm]$. Thus, given $R = r$, $\hat{\sigma}^2$ is distributed like $(\sigma^2/\Psi(r;\ \rho))\chi^2[mn]$. This result is the basis for the determination of the coverage probabilities and expected length.

The coverage probability, under σ^2 and ρ, of the intervals defined by (3.4) is

$$P_{\sigma^2,\rho}\{\hat{\sigma}^2/A_\gamma^*(R) \le \sigma^2 \le \hat{\sigma}^2/B_\gamma^*(R)\} = \qquad (4.5)$$

$$P_{\sigma^2,\rho}\{\sigma^2 B_\gamma^*(R) \le \hat{\sigma}^2 \le \sigma^2 A_\gamma^*(R)\}.$$

This coverage probability, CP, can be determined according to the previous result and the law of iterated expectations by

$$\begin{aligned} CP &= E\{P_{\sigma^2,\rho}\{\sigma^2 B_\gamma^*(R) \le \hat{\sigma}^2 \le \sigma^2 A_\gamma^*(R) \,|\, R\}\} \qquad (4.6) \\ &= E\{P_\rho\{\Psi(R;\ \rho) B_\gamma^*(R) \le \chi^2[mn] \le \Psi(R;\ \rho)\ A_\gamma^*(R) \,|\, R\}\}. \end{aligned}$$

Notice that the coverage probability is a function of ρ only.

Let $\mathrm{Pos}(k|\lambda)$ designate the c.d.f. of a Poisson distribution with mean λ. If mn is an even integer, the c.d.f. of the distribution of $\chi^2[nm]$ can be determined by $\mathrm{Pos}(k|\lambda)$ according to the well known relationship

$$P\{\chi^2[nm] \ge x\} = \mathrm{Pos}\ (nm/2 - 1 \mid x/2). \qquad (4.7)$$

Hence,

$$P\{\sigma^2 B_\gamma^*(R) \le \hat{\sigma}^2 \le \sigma^2\ A_\gamma^*(R) \mid R\} = \qquad (4.8)$$

$$\mathrm{Pos}\ (mn/2 - 1 \mid \lambda_1(r;\ \rho)) - \mathrm{Pos}\ (mn/2 - 1 \mid \lambda_2(r;\ \rho)),$$

where

$$\lambda_1(r;\ \rho) = \tfrac{1}{2}B_\gamma^{\ *}(r)\ \Psi(r;\ \rho)$$

and (4.9)

$$\lambda_2(r;\ \rho) = \tfrac{1}{2}A_\gamma^{\ *}(r)\ \Psi(r;\ \rho).$$

As shown earlier, the function $\varphi(R;\ \rho)$, given by (4.3), is distributed like $\beta(n/2,\ n(m-1)/2)$. Since $\varphi(R;\ \rho)$ is a strictly increasing function of R over the interval $(-(m-1)^{-1},1)$,

$$\begin{aligned} P_\rho\{r' \le R \le r''\} &= P_\rho\{\varphi(r';\ \rho) \le \varphi(R;\ \rho) \le \varphi(r'';\ \rho)\} \\ &= I_{\varphi(r'';\ \rho)}(n/2, n(m-1)/2) - \\ &\qquad I_{\varphi(r';\ \rho)}(n/2, n(m-1)/2), \end{aligned} \tag{4.10}$$

for any $-(m-1)^{-1} \le r' \le r'' \le 1$; where $I_a(p,q)$ is the incomplete beta function.

To evaluate the coverage probability (4.6) exactly one needs to integrate (4.8) with respect to the distribution of R over $(-(m-1)^{-1},1)$. We approximate this integral by partitioning the interval $(-(m-1)^{-1},1)$ into a large number, k, of subintervals (r_{i-1},r_i), $i=1,2,\ldots,k$; where $r_0 = -(m-1)^{-1}$; $r_k = 1$. The CP function is approximated then by

$$\begin{aligned} CP \cong \sum_{i=1}^{k} &[\mathrm{Pos}(mn/2 - 1 \mid \lambda_1(r_i^{\ *};\ \rho)) - \mathrm{Pos}(mn/2 - 1 \mid \lambda_2(r_i^{\ *};\ \rho))]. \\ &[I_{\varphi(r_i;\ \rho)}(n/2,n(m-1)/2) - I_{\varphi(r_{i-1};\rho)}(n/2,n(m-1)/2)] \end{aligned} \tag{4.11}$$

where $r_i^{\ *}$ is suitably chosen in (r_{i-1},r_i). In Table 1 and Table 2 we present the CP values of the MLE and Modified-MLE intervals. The

computations were performed with k=100 equal size subintervals with $r_i^* = (r_i + r_{i-1})/2$. The incomplete beta function was approximated using a Fourier series expansion (see Woods and Posten [10]).

The determination of the expected length of the MLE based intervals proceeds along similar lines. Let $L(R,\gamma)$ designate the length of such an interval having nominal coverage probability γ, i.e.

$$L(R,\gamma) = \hat{\sigma}^2 \left(\frac{1}{B_\gamma^*(R)} - \frac{1}{A_\gamma^*(R)} \right) . \tag{4.12}$$

The conditional expectation of $L(R,\gamma)$, given R, is

$$E\{L(R,\gamma) \mid R\} = \sigma^2 \frac{mn}{\Psi(R;\,\rho)} \left[\frac{1}{B_\gamma^*(R)} - \frac{1}{A_\gamma^*(R)} \right] . \tag{4.13}$$

Finally, the expected length of the MLE based intervals is approximated, similar to (4.11), by

$$E\{L(R,\gamma)\} = \sigma^2\, mn \sum_{i=1}^{k} \frac{A_\gamma^*(r_i^*) - B_\gamma^*(r_i^*)}{\Psi(r_i^*;\,\rho)\, B_\gamma^*(r_i^*) A_\gamma^*(r_i^*)} . \tag{4.14}$$

$$\left[I_{\varphi(r_i;\,\rho)}\left(\frac{n}{2}, \frac{n(m-1)}{2}\right) - I_{\varphi(r_{i-1};\,\rho)}\left(\frac{n}{2}, \frac{n(m-1)}{2}\right) \right]$$

Expected lengths of MLE based intervals for $\sigma^2 = 1$, $m = 2$, and varying values of n, ρ and γ are given in Table 3.

5. ESTIMATED -ρ INTERVALS

When ρ is known one can construct, on the basis of the statistic $Q(\rho)$, a system of UMAI confidence intervals for σ^2 at level $(1-\alpha)$. These intervals are of the form

$$\left(\frac{Q(\rho)}{\chi^2_{1-\epsilon_1}[mn]}, \frac{Q(\rho)}{\chi^2_{\epsilon_2}[mn]} \right) \tag{5.1}$$

where ϵ_1, ϵ_2 are determined so that $\epsilon_1 + \epsilon_2 = \alpha$ and an additional condition is satisfied (see Lehmann [4]). In practice, equal tail intervals are employed. Estimated $-\rho$ type intervals are obtained from (5.1) by substituting suitable estimators of ρ. If the MLE $\hat{\rho}$ is substituted, one obtains the UMAI confidence intervals for the case of $\rho = 0$ (known). As shown later, this results in a loss of coverage probability when $|\rho|$ is close to 1. Another approach is based on the Bonferroni inequality

$$P_{\rho,\sigma^2} \left\{ L_\alpha \frac{1-\rho\hat{\rho}}{1-\rho^2} \le \sigma^2 \le U_\alpha \frac{1-\rho\hat{\rho}}{1-\rho^2}, \; \underline{\rho}_\alpha \le \rho \le \bar{\rho}_\alpha \right\} \ge 1 - \alpha \tag{5.2}$$

for all (σ^2, ρ), where

$$L_\alpha = \frac{mn\hat{\sigma}^2}{\chi^2_{1-\alpha/4}[mn]}, \quad U_\alpha = \frac{mn\hat{\sigma}^2}{\chi^2_{\alpha/4}[mn]} \tag{5.3}$$

and $\underline{\rho}_\alpha$, $\bar{\rho}_\alpha$ are the lower and upper $(1-\alpha/2)$ confidence limits for ρ as obtained from (3.7). Consider the function

$$f(\rho\,;\hat{\rho}) = (1 - \rho\hat{\rho})/(1 - \rho^2), \quad -(m-1)^{-1} < \rho < 1. \tag{5.4}$$

The function $f(\rho\,;\hat{\rho})$ is convex in ρ, $\lim_{\rho\to 1} f(\rho\,;\hat{\rho}) = \infty$,

$$\lim_{\rho\to -(m-1)^{-1}} f(\rho\,;\hat{\rho}) = \begin{cases} \infty & m=2 \\ \dfrac{(m-1)(m-1+\hat{\rho})}{m(m-2)}, & m \ge 3 \end{cases}$$

and attains the minimum of $f_0(\hat{\rho}) = \frac{1}{2}(1 + (1-\hat{\rho}^2)^{\frac{1}{2}})$ at $\rho_0 = (1 - (1-\hat{\rho}^2)^{\frac{1}{2}})/\hat{\rho}$. Note that if $\hat{\rho} = 0$, $\rho_0 = 0$. Furthermore, if $\hat{\rho} \ge 0$ then $\rho_0 \le \hat{\rho}$, and vice versa. Inequality (5.2) prescribes a $(1-\alpha)$ level simultaneous

confidence region, $C_\alpha(\hat{\sigma}^2, \hat{\rho})$, for (σ^2, ρ). The boundaries of this region as functions of ρ are (see Figure 1)

$$\begin{aligned} b_1(\rho) &= L_\alpha f(\rho\ ;\ \hat{\rho}), \\ b_2(\rho) &= U_\alpha f(\rho\ ;\ \hat{\rho}), \\ b_3(\rho) &= \underline{\rho}_\alpha, \\ b_4(\rho) &= \bar{\rho}_\alpha. \end{aligned} \tag{5.5}$$

A confidence interval for σ^2, called the BF-interval, can be obtained by projecting $C_\alpha(\hat{\sigma}^2, \hat{\rho})$ on the σ^2-axis. This projection yields the upper confidence limit

$$\bar{\sigma}^2_\alpha = \begin{cases} U_\alpha\ f(\bar{\rho}_\alpha\ ;\ \hat{\rho}), & \hat{\rho} \geq 0 \\ U_\alpha\ f(\underline{\rho}_\alpha\ ;\ \hat{\rho}), & \hat{\rho} < 0 \end{cases} \tag{5.6}$$

and the lower confidence limit

$$\underline{\sigma}_\alpha^2 = L_\alpha\ g_\alpha\ (\hat{\rho}) \tag{5.7}$$

where

$$g_\alpha(\hat{\rho}) = \begin{cases} I\{\underline{\rho}_\alpha \geq \rho_o\}\ f(\underline{\rho}_\alpha\ ;\ \rho) + I\{\underline{\rho}_\alpha < \rho_o\} f_o(\hat{\rho}), & \hat{\rho} \geq 0 \\ I\{\bar{\rho}_\alpha \geq \rho_o\}\ f_o(\hat{\rho}) + I\{\bar{\rho}_\alpha < \rho_o\}\ f(\bar{\rho}_\alpha\ ;\ \hat{\rho}), & \hat{\rho} < 0 \end{cases} \tag{5.8}$$

and $I\{A\}$ is the indicator function of the set A. Notice that the intervals of σ^2 values, $(b_1(\hat{\rho}), b_2(\hat{\rho}))$ and $(b_1(0), b_2(0))$ coincide. Furthermore, the BF-interval $(\underline{\sigma}^2_\alpha, \bar{\sigma}^2_\alpha)$ contains $(b_1(0), b_2(0))$. The BF-intervals are, however, generally too conservative in the sense that their coverage probabilities are larger than $(1 - \alpha)$. Moreover, the BF-intervals could be considerably skewed around $\hat{\sigma}^2$ in cases of small n and $|\rho|$ close to 1. In Table 4 we present a few simulation estimates of

the coverage probabilities (CP) and the expected length (EL) of the (non-modified) MLE intervals and the BF-intervals for $1-\alpha = .90$, $m = 2$, $n = 10$ and several values of ρ. These simulation estimates are based on 100 independent and identical replicas for the case $\sigma^2 = 1$.

We remark that the CP and the EL of these intervals can be determined exactly according to the method discussed in the previous section. However, a small scale simulation was employed for the purpose of obtaining preliminary estimates only. The above estimates indicate that, indeed, the BF-interval provides higher coverage than desired and is very inefficient compared to the MLE intervals with respect to the expected length. An extensive evaluation of the relative efficiency of the MLE intervals will be given later.

6. NAIVE INTERVALS FOR σ^2 AT $m = 2$.

We discuss in the present section another attempt to derive a system of confidence intervals for σ^2, which may yield in small samples and for values of $|\rho|$ close to 1 more efficient results. The investigation is restricted to the case of $m = 2$. Accordingly, we consider n i.i.d. vectors (X_{1i}, X_{2i}), $i=1,2,\ldots,n$, having a joint bivariate normal distribution, with zero means, common variance σ^2 and correlation ρ.

Let $S_1 = \sum_{i=1}^{n} X_{1i}^2$, $S_2 = \sum_{i=1}^{n} X_{2i}^2$, and $S_{12} = \sum_{i=1}^{n} X_{1i} X_{2i}$.

(S_1+S_2, S_{12}) is a minimal sufficient statistic. A confidence interval for σ^2, at level $1-\alpha/2$, based on S_j only, $j = 1,2$ is given by $(S_j/\chi^2_{1-\alpha/4}[n],\ S_j/\chi^2_{\alpha/4}[n])$. From the Bonferroni inequality we imply that

the intersection (if not empty) of these two intervals is a confidence interval for σ^2 at level not less than $(1-\alpha)$. In other words, we consider the interval

$$\left(\frac{\max(S_1,S_2)}{\chi^2_{1-\alpha/4}[n]} \; , \; \frac{\min(S_1,S_2)}{\chi^2_{\alpha/4}[n]} \right) . \tag{6.1}$$

The interval (6.1) is empty, whenever $\max(S_1,S_2) \geq (\chi^2_{1-\alpha/4}[n]/\chi^2_{\alpha/4}[n]) \cdot \min(S_1,S_2)$. The probability of this event is generally negligible. Even in cases of $\rho \approx 0$ and $\alpha \approx .20$ the probability that (6.1) is empty does not exceed .01 for all n. (The computation of this probability is based on a formula given in Johnson & Kotz [2] p.222). For completeness we define the interval estimator when (6.1) is empty to be $(S_1/\chi^2_{1-\alpha/2}[n],\ S_1/\chi^2_{\alpha/2}[n])$. Since (6.1) is not a function of the minimal sufficient statistic its efficiency may be improved. There is reason to expect it to be efficient, however, when $|\rho| \approx 1$. We therefore develop a formula for determining the expected length of (6.1) in order to compare its efficiency with that of the MLE based intervals given by (3.4).

We develop an approximate formula for the expected length of the interval estimator, disregarding the event that (6.1) is empty. This approximation may lead to an error which is bounded by 10^{-2} in the cases under consideration.

Let $V = \min(S_1,S_2)$ and $U = \max(S_1,S_2)$. Since $U + V = S_1 + S_2$ the expected length of (6.1) is

$$EL = \frac{2n\,\sigma^2 - E\{U\}}{\chi^2_{\alpha/4}[n]} - \frac{E\{U\}}{\chi^2_{1-\alpha/4}[n]} \tag{6.2}$$

$$= \frac{2n\,\sigma^2}{\chi^2_{\alpha/4}[n]} - \left(\frac{1}{\chi^2_{\alpha/4}[n]} + \frac{1}{\chi^2_{1-\alpha/4}[n]} \right) E(U)$$

Let $F(\eta;\ \rho, \sigma^2)$ be the c.d.f. of U under (σ^2, ρ). It can be shown that (see Johnson & Kotz [2])

$$F(\eta;\ \rho, \sigma^2) = P_{\rho,\sigma^2}[S_1 \le \eta,\ \ S_2 \le \eta] \tag{6.3}$$

$$= \sum_{j=0}^{\infty} g(j|\rho^2, \tfrac{n}{2})\ [P(\chi^2[n + 2j] \le \eta|(1 - \rho^2))\ ]^2$$

where

$$g(j|\rho^2, \tfrac{n}{2}) = \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{n}{2})\ \Gamma(j + 1)}\ \rho^{2j}(1- \rho^2)^{n/2}, j = 0,1,\ldots \tag{6.4}$$

is the p.d.f. of a negative-binominal distribution. The expected value of U can be obtained by evaluating $\int_0^\infty [1 - F(\eta;\ \rho,\sigma^2)]\ d\eta$. After several manipulations one obtains for even sample size n the formula.

$$E(U) = 2n-2(1 - \rho^2) \sum_{j=0}^{\infty} g(j|\rho^2, \tfrac{n}{2}) \cdot \sum_{k=1}^{\frac{n}{2}+j} G(\tfrac{n}{2} + j -1 \mid \tfrac{1}{2}, k), \tag{6.5}$$

where $G(\cdot|\Psi,\nu)$ designates the c.d.f. corresponding to $g(\cdot|\Psi,\nu)$.

7. THE LOSS OF COVERAGE IN ASSUMING $\rho = 0$.

Suppose the investigator ignores the possibility of correlation. Under the assumption $\rho = 0$ a confidence interval for σ^2 could be obtained according to the distribution of $Q_1 + Q_2$ which is like that of $\sigma^2\chi^2[mn]$. The upper and lower limits, respectively, of a $1 - \alpha$ confidence interval are

$$\frac{Q_1 + Q_2}{\chi^2_{\alpha/2}[mn]} \quad \text{and} \quad \frac{Q_1 + Q_2}{\chi^2_{1-\alpha/2}[mn]}\,. \tag{7.1}$$

However, if the assumption $\rho = 0$ is incorrect the nominal coverage probability is unlikely to be attained. For the case where ρ is close to zero, actual coverage will be close to the nominal and thus the "cost of ignorance" is not great. But, for $|\rho|$ near unity actual coverage may be considerably less than the nominal resulting in substantial loss of coverage. In order to investigate the extent of loss in coverage probability when ρ is not zero we computed the actual coverage for $\rho = .1, .5,$ and $.9$ under varying nominal coverage, γ, and different m and n.

The actual coverage probability, AC, of the interval (7.1) is determined in a manner similar to that of determining the CP values of the MLE based intervals. Thus, for even mn,

$$AC = P_\rho\left\{\frac{Q_1 + Q_2}{\chi^2_{1-\alpha/2}[mn]} \le \sigma^2 \le \frac{Q_1 + Q_2}{\chi^2_{\alpha/2}[mn]}\right\} \tag{7.2}$$

$$= E\left\{P\left\{\frac{\Psi(R;\rho)\,\chi^2_{\alpha/2}[mn]}{mn} \le \chi^2[mn] \le \frac{\Psi(R;\rho)\,\chi^2_{1-\alpha/2}[mn]}{mn} \;\Big|\; R\right\}\right\}$$

$$\cong \sum_{i=1}^{k}\left[\mathrm{Pos}\left(\frac{mn}{2} - 1 \mid \eta_1(r_i^*;\rho)\right) - \mathrm{Pos}\left(\frac{mn}{2} - 1 \mid \eta_2(r_i^*;\rho)\right)\right] \cdot$$

$$\left[I_{\varphi(r_i;\,\rho)}\left(\frac{n}{2}, \frac{n(m-1)}{2}\right) - I_{\varphi(r_{i-1};\rho)}\left(\frac{n}{2}, \frac{n(m-1)}{2}\right)\right]$$

where

$$\eta_1(r;\rho) = \tfrac{1}{2}\,\frac{\Psi(r;\rho)}{mn}\,\chi^2_{\alpha/2}[mn] \quad \text{and}$$

$$\eta_2(r;\rho) = \tfrac{1}{2}\,\frac{\Psi(r;\rho)}{mn}\,\chi^2_{1-\alpha/2}[mn].$$

AC values of (7.1) are given in Table 5.

8. THE RELATIVE EFFICIENCY OF INTERVAL ESTIMATORS

The efficiency of interval estimators is a function of two components: their coverage probabilities (compared to the nominal ones) and their expected length. If ρ is known, the UMAI confidence intervals (5.1) are most efficient in the class of all location invariant and scale preserving confidence intervals, in the sense that among all such intervals having at least γ coverage the UMAI intervals have minimal expected length. However, when ρ is unknown UMAI confidence intervals do not exist. Asymptotically, for large samples, the MLE based intervals are most efficient in a wide class of intervals. However, as seen in the tables, for relatively small samples the MLE based intervals are not always better than other competitors. In order to compare alternative interval estimators, especially in small samples cases, we propose the following measure of relative efficiency

$$RE = \frac{E\{Q(\rho)\} \quad D(\tilde{\gamma}) \quad (1 + (m-1)\rho^2)^{\frac{1}{2}}}{EL}, \tag{8.1}$$

where $\tilde{\gamma}$ is the actual coverage probability of the interval estimator,

$$D(\tilde{\gamma}) = (\chi^2_{(1-\tilde{\gamma})/2}[nm])^{-1} - (\chi^2_{(1+\tilde{\gamma})/2}[nm])^{-1} \tag{8.2}$$

and EL is its expected length. Notice that $E\{Q(\rho)\}D(\tilde{\gamma})$ is the expected length of (5.1) having the same coverage probability, $\tilde{\gamma}$, as the interval estimator under consideration. Finally since

$$\lim_{n\to\infty} \frac{E\{L(R,\gamma)\}}{E\{Q(\rho)\}D(\gamma)} = (1 + (m-1)\rho^2)^{\frac{1}{2}} \tag{8.2}$$

the R.H.S. of (8.2) was introduced in the numerator of (8.1) in order

to provide a fair comparison. In Table 6 and Table 7 the relative efficiency values of the MLE and Modified MLE intervals are presented for various values of ρ, γ, n and m.

9. DISCUSSION AND CONCLUSION

In Table 1 we see that the coverage probabilities of the MLE intervals are uniformly close to the nominal ones. On the other hand, as shown in Table 2, the Modified-MLE intervals do exhibit deficient coverage, especially for large ρ values and small n. This deficiency is due to the modification which leads to shorter intervals. The loss in coverage that may result by ignoring the possibility of correlation is shown in Table 5 to be rather pronounced when ρ approaches 1 and m increases. For fixed m and ρ this loss of coverage is insensitive to n. The "naive" confidence interval (6.1) was designed to have coverage probability not less than the nominal one. It is therefore of interest to compare its expected length to that of the MLE estimators. This comparison is provided in Table 3. We see that for small n values and $\rho = .9$ the "naive" confidence interval may have a smaller expected length than the MLE interval. Such cases show areas of possible improvement over the MLE intervals. In Table 6 and Table 7 the relative efficiency, (8.1), of the MLE and Modified-MLE intervals is given. We see first that for small sample size (n=10) and large γ, the MLE interval may have infinite expected length. Furthermore, the relative efficiency of the Modified-MLE intervals is very close to 1 when $\rho = .1$ and decreases slowly as ρ approaches 1. The Modified-MLE intervals are considerably more efficient than the MLE intervals, despite

the fact that there is some loss in coverage.

In conclusion, confidence intervals ignoring the possibility of correlation should not be used. The estimated-ρ intervals discussed in Section 5 seem also to be inefficient. We do recommend the use of the Modified-MLE intervals (3.4). Although they may have some coverage deficiency in small samples these intervals are highly efficient. Finally, although the various tables present results for positive ρ values, similar results would be obtained for negative values of ρ.

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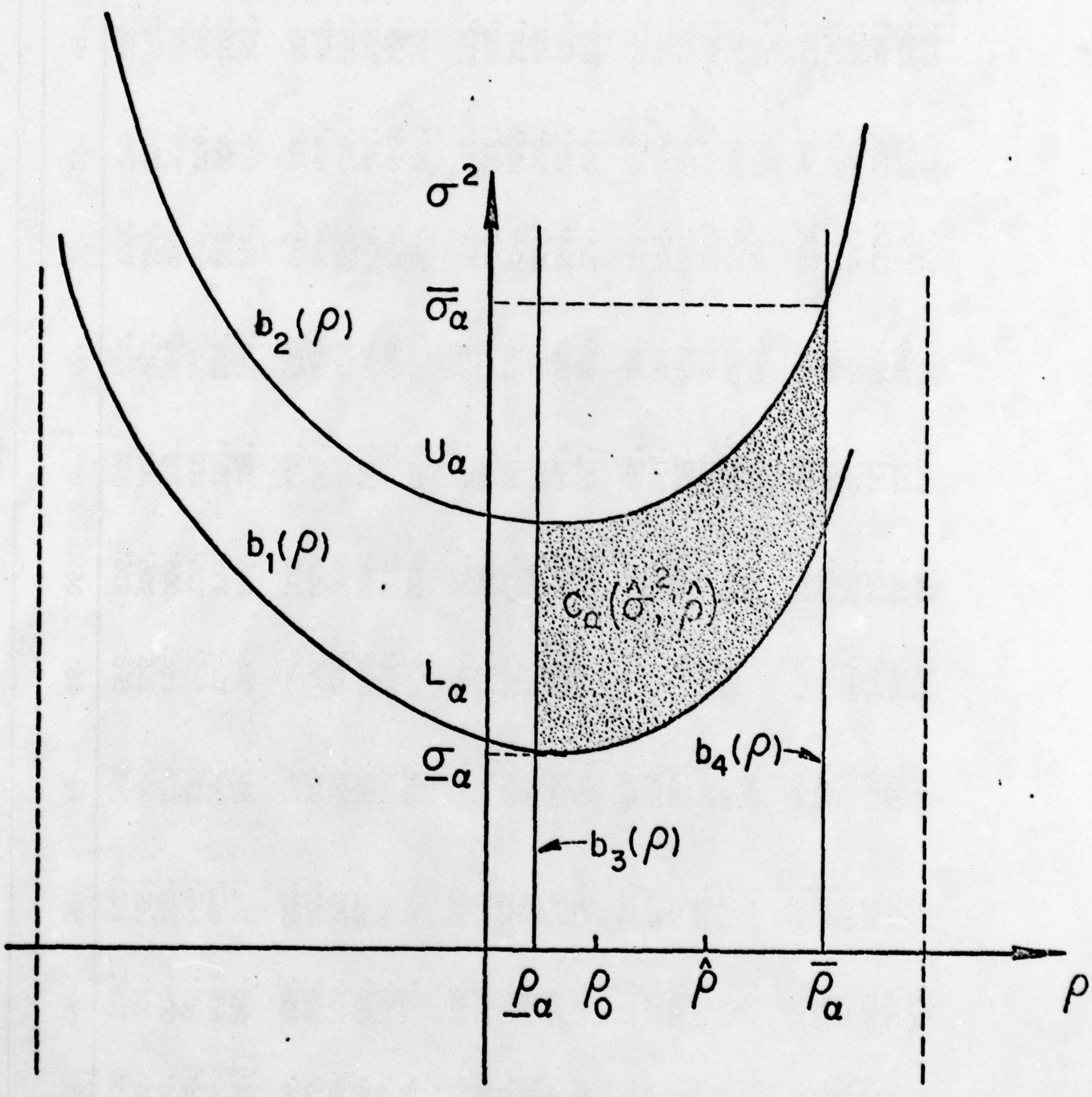

Figure 1 - Boundaries for simultaneous confidence region, $C_\alpha(\hat{\sigma}^2, \hat{\rho})$, for (σ^2, ρ) for case of $m=2$ and $\hat{\rho} \geq 0$.

TABLE 1 - Coverage probabilities of MLE intervals.

	ρ	.10				.50				.90			
m	γ	.80	.90	.95	.99	.80	.90	.95	.99	.80	.90	.95	.99
	10	.830	.923	.961	.988	.818	.916	.959	.988	.814	.911	.952	.983
	20	.816	.912	.957	.985	.807	.906	.952	.986	.805	.904	.949	.984
	30	.808	.904	.950	.986	.802	.900	.949	.987	.801	.900	.948	.985
2	40	.809	.903	.949	.987	.802	.901	.949	.987	.799	.899	.947	.986
	50	.807	.906	.953	.988	.803	.902	.951	.988	.799	.899	.948	.986
	60	.806	.905	.952	.988	.802	.902	.951	.989	.797	.898	.947	.987
	10	.822	.917	.959	.989	.807	.907	.955	.990	.809	.907	.950	.983
	20	.808	.904	.950	.987	.800	.899	.948	.988	.802	.902	.948	.985
	30	.808	.907	.953	.987	.801	.900	.949	.988	.801	.900	.948	.986
3	40	.806	.905	.952	.989	.800	.900	.950	.989	.799	.899	.948	.987
	50	.805	.904	.952	.989	.800	.900	.950	.989	.798	.898	.948	.988
	60	.804	.903	.952	.989	.800	.900	.950	.989	.796	.897	.947	.987
	10	.818	.914	.957	.986	.799	.899	.949	.988	.806	.903	.948	.983
	20	.809	.905	.950	.987	.797	.897	.947	.988	.802	.901	.948	.985
	30	.807	.905	.953	.989	.799	.898	.948	.989	.800	.900	.948	.986
4	40	.805	.904	.952	.989	.799	.899	.949	.989	.798	.899	.948	.987
	50	.804	.903	.952	.989	.799	.899	.949	.989	.797	.897	.947	.987
	60	.803	.903	.951	.989	.799	.899	.949	.989	.796	.896	.947	.987
	10	.812	.907	.952	.987	.792	.893	.945	.988	.804	.902	.947	.982
	20	.808	.907	.953	.989	.796	.896	.946	.988	.802	.901	.948	.985
	30	.806	.905	.952	.989	.797	.897	.947	.989	.800	.900	.948	.986
5	40	.804	.903	.952	.989	.798	.897	.948	.989	.798	.898	.948	.987
	50	.804	.903	.952	.990	.798	.898	.948	.989	.797	.897	.947	.987
	60	.803	.902	.951	.990	.798	.898	.948	.989	.795	.896	.946	.987
	10	.811	.906	.952	.987	.789	.890	.942	.987	.804	.902	.946	.982
	20	.807	.906	.953	.989	.794	.894	.945	.988	.802	.901	.948	.985
	30	.805	.904	.952	.989	.796	.896	.946	.988	.799	.899	.948	.986
6	40	.804	.903	.952	.990	.797	.897	.947	.988	.798	.898	.947	.987
	50	.803	.902	.951	.990	.797	.897	.947	.989	.796	.897	.947	.987
	60	.803	.902	.951	.990	.798	.897	.948	.989	.795	.896	.946	.987

TABLE 2 - Coverage probabilities of Modified-MLE intervals.

m	p	.10				.50				.90			
	n \ γ	.80	.90	.95	.99	.80	.90	.95	.99	.80	.90	.95	.99
2	10	.816	.886	.921	.962	.804	.868	.904	.951	.793	.852	.889	.939
	20	.818	.899	.939	.977	.810	.891	.929	.970	.810	.886	.923	.967
	30	.813	.904	.946	.982	.810	.899	.939	.978	.813	.898	.937	.976
	40	.815	.910	.953	.985	.812	.904	.946	.982	.814	.905	.944	.982
	50	.815	.912	.957	.990	.814	.908	.951	.986	.817	.909	.950	.986
	60	.815	.913	.958	.991	.814	.910	.954	.988	.817	.911	.953	.988
3	10	.815	.893	.929	.968	.800	.867	.903	.949	.790	.850	.886	.936
	20	.809	.900	.942	.979	.804	.888	.927	.968	.806	.883	.920	.964
	30	.810	.906	.952	.987	.807	.897	.938	.977	.811	.896	.934	.975
	40	.810	.908	.954	.989	.808	.901	.944	.981	.812	.902	.942	.980
	50	.810	.908	.955	.990	.809	.903	.947	.983	.813	.905	.947	.983
	60	.810	.909	.955	.991	.809	.905	.949	.985	.813	.907	.950	.985
4	10	.813	.893	.933	.972	.795	.865	.901	.947	.788	.848	.884	.933
	20	.808	.903	.947	.981	.801	.886	.925	.966	.803	.881	.917	.961
	30	.808	.905	.951	.988	.805	.895	.936	.975	.808	.894	.932	.972
	40	.808	.906	.953	.989	.806	.899	.941	.979	.810	.899	.940	.978
	50	.808	.906	.954	.990	.807	.901	.945	.982	.810	.902	.944	.981
	60	.808	.907	.954	.990	.807	.902	.947	.983	.810	.904	.947	.983
5	10	.807	.895	.936	.974	.793	.862	.898	.945	.786	.846	.882	.930
	20	.807	.903	.946	.986	.801	.885	.924	.965	.802	.880	.916	.960
	30	.807	.904	.946	.988	.803	.893	.934	.973	.806	.892	.930	.971
	40	.807	.905	.946	.989	.804	.897	.939	.977	.807	.897	.938	.976
	50	.807	.905	.953	.990	.805	.899	.943	.980	.808	.900	.942	.980
	60	.807	.906	.953	.990	.805	.900	.945	.982	.807	.902	.945	.982
6	10	.807	.896	.938	.976	.791	.860	.896	.943	.784	.845	.880	.928
	20	.807	.903	.949	.986	.802	.884	.922	.964	.801	.878	.915	.958
	30	.806	.904	.951	.988	.802	.891	.932	.972	.804	.890	.929	.969
	40	.806	.904	.952	.989	.803	.895	.938	.976	.806	.895	.936	.975
	50	.806	.905	.953	.990	.804	.898	.941	.979	.806	.899	.941	.979
	60	.806	.905	.953	.990	.804	.899	.944	.981	.806	.900	.944	.981

TABLE 3 - Expected length of the "Naive" and MLE-based intervals for $m=2$, $\sigma^2=1$.

γ	n \ ρ	.10			.50			.90		
		Naive	MLE	M-MLE[a]	Naive	MLE	M-MLE	Naive	MLE	M-MLE
	10	1.241	1.038	.879	1.339	1.181	.960	1.665	1.559	1.162
	20	.772	.645	.610	.829	.723	.673	1.014	.907	.821
.80	30	.605	.506	.495	.648	.566	.548	.790	.700	.671
	40	.514	.430	.427	.550	.480	.474	.669	.590	.581
	50	.454	.381	.381	.486	.424	.423	.591	.519	.519
	60	.411	.345	.347	.440	.384	.386	.534	.469	.474
	10	1.736	1.559	1.145	1.850	1.858	1.253	2.230	2.769	1.531
	20	1.035	.883	.795	1.096	1.006	.878	1.295	1.314	1.082
.90	30	.801	.677	.644	.846	.764	.715	.995	.967	.883
	40	.676	.569	.556	.713	.639	.618	.837	.799	.765
	50	.595	.500	.496	.627	.561	.552	.735	.695	.684
	60	.538	.451	.452	.566	.505	.504	.663	.623	.625
	10	2.282	2.277	1.386	2.415	2.948	1.520	2.858	5.634	1.876
	20	1.296	1.131	.962	1.361	1.316	1.065	1.577	1.814	1.326
.95	30	.988	.843	.780	1.036	.963	.867	1.193	1.256	1.083
	40	.828	.700	.673	.867	.793	.750	.995	1.011	.938
	50	.726	.611	.601	.760	.689	.670	.871	.868	.839
	60	.654	.549	.547	.684	.618	.611	.783	.771	.765
	10	3.859	-[b]	1.888	4.052	-	2.208	4.695	-	2.325
	20	1.924	1.828	1.315	2.003	2.288	1.463	2.263	3.914	1.871
.99	30	1.413	1.248	1.066	1.467	1.476	1.191	1.645	2.116	1.528
	40	1.163	1.000	.920	1.206	1.160	1.030	1.347	1.569	1.323
	50	1.009	.856	.821	1.046	.983	.920	1.166	1.292	1.183
	60	.903	.760	.748	.935	.867	.839	1.041	1.120	1.080

a. M-MLE designates Modified MLE.
b. A dash ,-, indicates infinite expected length.

TABLE 4 - Simulation estimates of coverage probability (CP) and expected length (EL) of the MLE and BF intervals for $1-\alpha = .90$, $n=10$, $m=2$, 100 replicas.

	$\rho=.10$		$\rho=.50$		$\rho=.90$	
Interval	CP	EL	CP	EL	CP	EL
MLE	.91	1.574	.90	1.871	.88	2.799
BF	.95	2.599	.95	2.996	.96	3.658

TABLE 5 - Coverage probabilities of confidence intervals assuming ρ=0.

	ρ	.10				.50				.90			
m	γ / n	.80	.90	.95	.99	.80	.90	.95	.99	.80	.90	.95	.99
	10	.798	.899	.949	.990	.752	.862	.924	.981	.657	.777	.855	.946
	20	.798	.898	.947	.987	.749	.860	.920	.977	.656	.775	.852	.943
	30	.794	.893	.944	.987	.748	.857	.918	.977	.655	.774	.850	.941
2	40	.798	.897	.945	.988	.748	.858	.919	.977	.654	.773	.849	.941
	50	.798	.899	.949	.990	.749	.860	.921	.979	.654	.774	.851	.942
	60	.798	.899	.949	.990	.749	.859	.921	.979	.653	.773	.850	.941
	10	.796	.897	.948	.989	.710	.826	.896	.968	.568	.687	.772	.889
	20	.792	.892	.943	.987	.706	.821	.890	.964	.567	.687	.770	.886
	30	.796	.897	.948	.988	.706	.823	.892	.965	.568	.687	.770	.886
3	40	.796	.897	.948	.989	.706	.822	.892	.965	.567	.686	.770	.885
	50	.796	.897	.948	.989	.705	.822	.892	.965	.566	.685	.769	.884
	60	.796	.897	.948	.989	.705	.822	.891	.965	.565	.684	.768	.884
	10	.794	.895	.945	.986	.672	.792	.867	.952	.506	.621	.706	.834
	20	.793	.894	.943	.987	.669	.788	.863	.950	.507	.621	.706	.833
	30	.793	.895	.947	.989	.669	.788	.864	.950	.507	.621	.706	.833
4	40	.793	.895	.947	.989	.668	.788	.863	.950	.506	.621	.705	.832
	50	.793	.895	.947	.989	.668	.787	.863	.950	.506	.620	.704	.831
	60	.793	.895	.947	.989	.668	.787	.863	.949	.505	.619	.703	.830
	10	.788	.889	.941	.986	.638	.760	.840	.936	.461	.570	.653	.786
	20	.791	.894	.946	.988	.637	.759	.838	.935	.463	.572	.655	.786
	30	.791	.894	.946	.988	.637	.757	.837	.934	.462	.571	.655	.785
5	40	.791	.894	.946	.988	.636	.757	.836	.933	.462	.571	.654	.784
	50	.791	.893	.946	.988	.636	.756	.836	.933	.461	.570	.653	.783
	60	.791	.893	.946	.988	.636	.756	.835	.932	.460	.569	.652	.782
	10	.787	.887	.940	.986	.611	.733	.816	.920	.427	.530	.612	.744
	20	.789	.892	.945	.988	.609	.731	.813	.918	.428	.532	.613	.745
	30	.789	.892	.945	.988	.608	.729	.811	.917	.428	.531	.612	.744
6	40	.789	.892	.944	.988	.608	.729	.811	.916	.427	.531	.612	.743
	50	.789	.892	.944	.988	.608	.728	.810	.916	.426	.530	.611	.742
	60	.789	.892	.944	.988	.607	.728	.810	.915	.426	.529	.610	.741

TABLE 6 - Relative efficiency of MLE intervals.

	ρ	.10				.50				.90			-
m	n \ γ	.80	.90	.95	.99	.80	.90	.95	.99	.80	.90	.95	.99
2	10	.945	.849	.712	-[a]	.893	.771	.602	-	.807	.611	.364	-
	20	.978	.940	.887	.684	.950	.897	.829	.620	.906	.821	.713	.424
	30	.979	.948	.909	.794	.962	.924	.880	.752	.934	.878	.809	.618
	40	.991	.963	.931	.845	.972	.945	.911	.815	.948	.905	.855	.712
	50	.993	.980	.959	.884	.981	.962	.937	.858	.957	.925	.885	.767
	60	.995	.984	.967	.904	.984	.968	.948	.884	.960	.934	.901	.803
3	10	.964	.908	.830	-	.884	.786	.664	-	.780	.589	.362	-
	20	.977	.946	.906	.790	.940	.893	.839	.690	.890	.804	.700	.429
	30	.992	.976	.952	.860	.962	.933	.899	.802	.927	.870	.803	.620
	40	.994	.983	.966	.903	.972	.951	.926	.855	.942	.900	.850	.712
	50	.995	.987	.973	.924	.978	.961	.941	.885	.950	.917	.877	.766
	60	.996	.989	.978	.937	.981	.968	.951	.905	.954	.927	.894	.801
4	10	.973	.932	.875	-	.875	.784	.677	-	.765	.577	.360	-
	20	.988	.962	.928	.841	.936	.894	.846	.722	.885	.797	.695	.431
	30	.993	.982	.965	.903	.960	.933	.902	.822	.921	.865	.798	.619
	40	.995	.987	.974	.928	.970	.950	.927	.868	.937	.895	.846	.710
	50	.996	.990	.980	.943	.976	.960	.942	.895	.946	.913	.873	.764
	60	.997	.991	.983	.953	.980	.966	.952	.912	.951	.923	.890	.798
5	10	.968	.929	.880	-	.866	.780	.682	-	.757	.571	.358	-
	20	.990	.977	.955	.880	.937	.896	.852	.739	.881	.794	.693	.433
	30	.994	.985	.972	.923	.958	.931	.902	.829	.918	.861	.795	.618
	40	.996	.989	.979	.943	.968	.949	.927	.873	.935	.893	.843	.709
	50	.997	.991	.984	.955	.975	.959	.942	.898	.944	.910	.870	.762
	60	.997	.993	.986	.963	.979	.965	.951	.915	.949	.921	.888	.797
6	10	.972	.939	.898	-	.864	.780	.687	-	.753	.569	.358	-
	20	.991	.980	.962	.901	.934	.894	.851	.744	.878	.791	.690	.433
	30	.994	.987	.976	.936	.956	.930	.902	.831	.916	.859	.792	.617
	40	.996	.991	.982	.953	.967	.947	.926	.874	.933	.890	.841	.708
	50	.997	.993	.986	.963	.974	.958	.941	.899	.943	.909	.868	.760
	60	.997	.994	.989	.969	.978	.965	.951	.916	.947	.920	.887	.795

a. A dash, - , indicates zero efficiency, due to infinite expected length.

TABLE 7 - Relative efficiency of Modified-MLE intervals.

	ρ	.10				.50				.90			
m	n \ γ	.80	.90	.95	.99	.80	.90	.95	.99	.80	.90	.95	.99
2	10	1.075[a]	1.011	.947	.862	1.060	.971	.902	.832	1.027	.912	.834	.823
	20	1.040	.999	.959	.875	1.030	.979	.922	.829	1.014	.942	.871	.763
	30	1.015	.997	.966	.891	1.014	.983	.937	.848	1.002	.955	.898	.788
	40	1.014	1.008	.982	.903	1.009	.988	.951	.866	.996	.963	.909	.809
	50	1.011	1.007	.995	.950	1.009	.995	.967	.896	.997	.970	.926	.831
	60	1.009	1.006	.996	.955	1.007	.996	.972	.905	.994	.971	.932	.843
3	10	1.052	1.011	.955	.871	1.030	.944	.876	.786	.993	.880	.801	.768
	20	1.015	.998	.968	.896	1.010	.967	.912	.820	.996	.926	.853	.744
	30	1.012	1.008	.995	.949	1.007	.981	.939	.854	.995	.947	.884	.780
	40	1.009	1.006	.996	.958	1.006	.986	.952	.874	.993	.957	.904	.803
	50	1.007	1.005	.997	.964	1.004	.989	.960	.887	.990	.961	.916	.820
	60	1.006	1.004	.998	.968	1.004	.990	.966	.898	.988	.964	.924	.832
4	10	1.039	.998	.960	.880	1.008	.928	.860	.767	.978	.866	.788	.742
	20	1.013	1.006	.985	.910	1.002	.959	.904	.812	.987	.917	.846	.738
	30	1.009	1.006	.996	.958	1.003	.975	.932	.847	.990	.942	.880	.776
	40	1.007	1.004	.997	.965	1.003	.981	.946	.867	.989	.952	.900	.800
	50	1.005	1.004	.998	.971	1.002	.985	.955	.881	.987	.958	.913	.817
	60	1.005	1.003	.998	.974	1.002	.987	.962	.892	.985	.961	.921	.830
5	10	1.018	.996	.963	.888	.997	.917	.848	.754	.969	.859	.781	.728
	20	1.010	1.006	.993	.951	1.000	.957	.901	.810	.986	.917	.845	.737
	30	1.007	1.004	.996	.962	1.001	.971	.927	.841	.987	.939	.878	.774
	40	1.005	1.003	.997	.970	1.001	.978	.942	.861	.987	.950	.898	.799
	50	1.004	1.003	.998	.975	1.000	.982	.952	.877	.985	.956	.912	.817
	60	1.004	1.002	.998	.978	1.000	.985	.959	.889	.983	.959	.921	.831
6	10	1.015	.995	.965	.894	.989	.907	.838	.744	.964	.854	.776	.718
	20	1.008	1.005	.993	.954	.998	.952	.895	.803	.983	.914	.843	.736
	30	1.006	1.003	.996	.966	.999	.968	.923	.835	.985	.938	.877	.774
	40	1.004	1.003	.997	.973	.999	.976	.939	.857	.985	.949	.898	.799
	50	1.003	1.002	.998	.977	.999	.981	.949	.873	.984	.955	.911	.817
	60	1.003	1.002	.998	.980	.999	.984	.957	.886	.982	.958	.921	.832

a. Values of RE may exceed 1 owing to the definition of RE as well as computing error.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Confidence intervals for the common variance of equicorrelated normal random variable are investigated. When the correlation coefficient is unknown no uniformly most accurate invariant confidence interval for the common variance exists. We develop interval estimators which are based on the asymptotically most efficient ones by employing the best asymptotic normality of the maximum likelihood estimators. In addition to their comparison with possible competitors, their small sample efficiency is

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studied. A method for determining exact coverage probabilities and expected length is developed. As a result of these investigations we conclude that interval estimators based on maximum likelihood estimators are to be recommended.